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METHODS FOR DETERMINATION AND APPROXIMATION OF DOMAINS OF ATTRACTION IN THE CASE OF AUTONOMOUS DISCRETE DYNAMICAL SYSTEMS

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Abstract. A method for determination and two methods for approximation of the domain of attraction $D_a(0)$ of an asymptotically stable steady state of an autonomous, $\mathbb{R}$-analytical, discrete system is presented. The method of determination is based on the construction of a Lyapunov function V, whose domain of analyticity is $D_a(0)$. The first method of approximation uses a sequence of Lyapunov functions V_p, which converges to the Lyapunov function V on $D_a(0)$. Each V_p defines an estimate N_p of $D_a(0)$. For any $x \in D_a(0)$ there exists an estimate N_{p^x} which contains x. The second method of approximation uses a ball $B(R) \subset D_a(0)$ which generates the sequence of estimates $M_p = f^{-p}(B(R))$. For any $x \in D_a(0)$ there exists an estimate M_{p^x} which contains x. The cases $\|\partial_0 f\| < 1$ and $\rho(\partial_0 f) < 1$ are treated separately (even though the second case includes the first one) because significant differences occur.

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1. **Introduction.** Let be the following discrete dynamical system:

$$x_{k+1} = f(x_k) \qquad k = 0, 1, 2... \tag{1.1}$$

where $f : \Omega \to \Omega$ is an $\mathbb{R}$-analytic function defined on a domain $\Omega \subset \mathbb{R}^n$, $0 \in \Omega$ and $f(0) = 0$, i.e. $x = 0$ is a steady state (fixed point) of (1.1).

For $r > 0$, denote by $B(r) = \{x \in \mathbb{R}^n : \|x\| < r\}$ the ball of radius r.

The steady state $x = 0$ of (1.1) is "stable" provided that given any ball $B(\varepsilon)$, there is a ball $B(\delta)$ such that if $x \in B(\delta)$ then $f^k(x) \in B(\varepsilon)$, for $k = 0, 1, 2, ...$ [1].

If in addition there is a ball $B(r)$ such that $f^k(x) \to 0$ as $k \to \infty$ for all $x \in B(r)$ then the steady state $x = 0$ is "asymptotically stable" [1].

The domain of attraction $D_a(0)$ of the asymptotically stable steady state $x = 0$ is the set of initial states $x \in \Omega$ from which the system converges to the steady state itself i.e.

$$D_a(0) = \{x \in \Omega | f^k(x) \stackrel{k\to\infty}{\longrightarrow} 0\} \tag{1.2}$$

Theoretical research shows that the $D_a(0)$ and its boundary are complicated sets. In most cases, they do not admit an explicit elementary representation. The domain of attraction of an asymptotically stable steady state of a discrete dynamical system is not necessarily connected (which is the case for continuous dynamical systems). This fact is shown by the following example.

Example 1.1. Let be the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4$. The domain of attraction of the asymptotically stable steady state $x = 0$ is $D_a(0) = (-2.79, -2.46) \cup (-1, 1)$ which is not connected. □

Different procedures are used for the approximation of the $D_a(0)$ with domains having a simpler shape. For example, in the case of the Theorem 4.20 pg. 170 [1] the domain which approximates the $D_a(0)$ is defined by a Lyapunov function V built with the matrix $\partial_0 f$ of the linearized system in 0, under the assumption $\|\partial_0 f\| < 1$. In [2], a Lyapunov function V is presented in the case when the matrix $\partial_0 f$ is a contraction, i.e. $\|\partial_0 f\| < 1$. The Lyapunov function V is built using the whole nonlinear system, not only the matrix $\partial_0 f$. V is defined on the whole $D_a(0)$, and more, the $D_a(0)$ is the natural domain of analyticity of V. In [3], this result is extended for the more general case when $\rho(\partial_0 f) < 1$ (where $\rho(\partial_0 f)$ denotes the spectral radius of $\partial_0 f$). This last result is the following:

Theorem 1.1. *(see [3]) If the function f satisfies the following conditions:*

$$f(0) = 0 \tag{1.3}$$

$$\rho(\partial_0 f) < 1 \tag{1.4}$$

then 0 is an asymptotically stable steady state. $D_a(0)$ is an open subset of Ω and coincides with the natural domain of analyticity of the unique solution V of the iterative first order functional equation

$$\begin{cases} V(f(x)) - V(x) = -\|x\|^2 \\ V(0) = 0 \end{cases} \tag{1.5}$$

The function V is positive on $D_a(0)$ and $V(x) \stackrel{x\to x^0}{\longrightarrow} +\infty$, for any $x^0 \in \partial D_a(0)$ ($\partial D_a(0)$ denotes the boundary of $D_a(0)$) or for $\|x\| \to \infty$.

The function V is given by

$$V(x)=\sum_{k=0}^{\infty}\|f^k(x)\|^2 \qquad \textit{for any } x\in D_a(0) \tag{1.6}$$

The Lyapunov function V can be found theoretically using relation (1.6). In the followings, we will shortly present the procedure of determination and approximation of the domain of attraction using the function V presented in [2, 3].

The region of convergence D_0 of the power series development of V in 0 is a part of the domain of attraction $D_a(0)$. If D_0 is strictly contained in $D_a(0)$, then there exists a point $x^0\in\partial D_0$ such that the function V is bounded on a neighborhood of x^0. Let be the power series development of V in x^0. The domain of convergence D_1 of the series centered in x^0 gives a new part $D_1\setminus(D_0\bigcap D_1)$ of the domain of attraction $D_a(0)$. At this step, the part $D_0\bigcup D_1$ of $D_a(0)$ is obtained.

If there exists a point $x^1\in\partial(D_0\bigcup D_1)$ such that the function V is bounded on a neighborhood of x^1, then the domain $D_0\bigcup D_1$ is strictly included in the domain of attraction $D_a(0)$. In this case, the procedure described above is repeated in x^1.

The procedure cannot be continued in the case when it is found that on the boundary of the domain $D_0\bigcup D_1\bigcup\ldots\bigcup D_p$ obtained at step p, there are no points having neighborhoods on which V is bounded.

This procedure gives an open connected estimate D of the domain of attraction $D_a(0)$. Note that $f^{-k}(D)$, $k\in\mathbb{N}$ is also an estimate of $D_a(0)$, which is not necessarily connected.

The procedure described above is illustrated by the following examples:

Example 1.2. Let be the $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^2$. Due to the equality $f^k(x)=x^{2^k}$ the domain of attraction of the asymptotically stable steady state $x=0$ is $D_a(0)=(-1,1)$. The Lyapunov function is $V(x)=\sum\limits_{k=0}^{\infty}x^{2^{k+1}}$. The domain of convergence of the series is $D_0=(-1,1)$ which coincides with $D_a(0)$. □

Example 1.3. Let be the function $f:\Omega=(-\infty,1)\to\Omega$ defined by $f(x)=\frac{x}{e+(1-e)x}$. Due to the equality $f^k(x)=\frac{x}{e^k+(1-e^k)x}$ the domain of attraction of the asymptotically stable steady state $x=0$ is $D_a(0)=(-\infty,1)$. The power series expansion of the Lyapunov function $V(x)=\sum\limits_{k=0}^{\infty}|f^k(x)|^2$ in 0 is

$$V(x)=\sum_{m=2}^{\infty}(m-1)\sum_{k=0}^{\infty}e^{-2k}(1-e^{-k})^{m-2}x^m \tag{1.7}$$

The radius of convergence of the series (1.7) is

$$r_0=\lim_{m\to\infty}\sqrt[m]{(m-1)\sum_{k=0}^{\infty}e^{-2k}(1-e^{-k})^{m-2}}=1 \tag{1.8}$$

therefore the domain of convergence of the series (1.7) is $D_0=(-1,1)\subset D_a(0)$. More, $V(x)\to\infty$ as $x\to1$ and $V(-1)<\infty$. The radius of convergence of the power series expansion of V in -1 is

$$r_{-1}=\lim_{m\to\infty}\sqrt[m]{\sum_{k=1}^{\infty}\frac{e^k(e^k-1)^{m-2}[(m-3)e^k+2]}{(2e^k-1)^{m+2}}}=1 \tag{1.9}$$

therefore, the domain of convergence of the power series development of V in -1 is $D_{-1} = (-2, 0)$ which gives a new part of $D_a(0)$. □

Numerical results for more complex examples are given in [2, 3].

2. **Theoretical results when matrix $A = \partial_0 f$ is a contraction, i.e. $\|A\| < 1$.** The function f can be written as

$$f(x) = Ax + g(x) \qquad \text{for any } x \in \Omega \tag{2.10}$$

where $A = \partial_0 f$ and $g : \Omega \to \Omega$ is an $\mathbb{R}$-analytic function such that $g(0) = 0$ and $\lim\limits_{x\to 0} \frac{\|g(x)\|}{\|x\|} = 0$.

Proposition 2.1. *If $\|A\| < 1$, then there exists $r > 0$ such that $B(r) \subset \Omega$ and $\|f(x)\| < \|x\|$ for any $x \in B(r) \setminus \{0\}$.*

Proof. Due to the fact that $\lim\limits_{x\to 0} \frac{\|g(x)\|}{\|x\|} = 0$ there exists $r > 0$ such that $B(r) \subset \Omega$ and

$$\|g(x)\| < (1 - \|A\|)\|x\| \qquad \text{for any } x \in B(r) \setminus \{0\} \tag{2.11}$$

Let be $x \in B(r) \setminus \{0\}$. Inequality (2.11) provides that

$$\|f(x)\| = \|Ax + g(x)\| \le \|A\|\|x\| + \|g(x)\| < (\|A\| + 1 - \|A\|)\|x\| = \|x\| \tag{2.12}$$

therefore, $\|f(x)\| < \|x\|$. □

Definition 2.1. *Let be $R > 0$ the largest number such that $B(R) \subset \Omega$ and $\|f(x)\| < \|x\|$ for any $x \in B(R) \setminus \{0\}$.*

If for any $r > 0$ we have that $B(r) \subset \Omega$ and $\|f(x)\| < \|x\|$ for any $x \in B(r) \setminus \{0\}$, then $R = +\infty$ and $B(R) = \Omega = \mathbb{R}^n$.

Lemma 2.2. a. *$B(R)$ is invariant to the flow of system (1.1).*

b. *For any $x \in B(R)$, the sequence $(\|f^k(x)\|)_{k\in\mathbb{N}}$ is decreasing.*

c. *For any $p \ge 0$ and $x \in B(R) \setminus \{0\}$, $\Delta V_p(x) = V_p(f(x)) - V_p(x) < 0$, where*

$$V_p(x) = \sum_{k=0}^{p} \|f^k(x)\|^2 \qquad \textit{for } x \in \Omega \tag{2.13}$$

Proof. a. If $x = 0$, then $f^k(0) = 0$, for any $k \in \mathbb{N}$. For $x \in B(R) \setminus \{0\}$, we have $\|f(x)\| < \|x\|$, which implies that $f(x) \in B(R)$, i.e. $B(R)$ is invariant to the flow of system (1.1).

b. By induction, it results that for $x \in B(R)$ we have $f^k(x) \in B(R)$ and $\|f^{k+1}(x)\| \le \|f^k(x)\|$, which means that the sequence $(\|f^k(x)\|)_{k\in\mathbb{N}}$ is decreasing.

c. In particular, for $p \ge 0$ and $x \in B(R)$, we have $\|f^{p+1}(x)\| \le \|f(x)\| < \|x\|$ and therefore, $\Delta V_p(x) = \|f^{p+1}(x)\|^2 - \|x\|^2 < 0$.

□

Corollary 2.3. *For any $p \ge 0$, there exists a maximal domain $G_p \subset \Omega$ such that $0 \in G_p$ and for $x \in G_p \setminus \{0\}$, the (positive definite) function V_p verifies $\Delta V_p(x) < 0$. In other words, for any $p \ge 0$ the function V_p defined by (2.13) is a Lyapunov function for (1.1) on G_p. More, $B(R) \subset G_p$ for any $p \ge 0$.*

Theorem 2.4. *$B(R)$ is an invariant set included in the domain of attraction $D_a(0)$.*

Proof. Let be $x \in B(R) \setminus \{0\}$. We have to prove that $\lim_{k\to\infty} f^k(x) = 0$.

The sequence $(f^k(x))_{k\in\mathbb{N}}$ is bounded: $f^k(x)$ belongs to $B(R)$. Let be $(f^{k_j}(x))_{j\in\mathbb{N}}$ a convergent subsequence and let be $\lim_{j\to\infty} f^{k_j}(x) = y^0$. It is clear that $y^0 \in B(R)$.

It can be shown that

$$\|f^k(x)\| \geq \|y^0\| \qquad \text{for any } k \in \mathbb{N} \tag{2.14}$$

For this, observe first that $f^{k_j}(x) \to y^0$ and $(\|f^{k_j}(x)\|)_{k\in\mathbb{N}}$ is decreasing (Lemma 2.2). These imply that $\|f^{k_j}(x)\| \geq \|y^0\|$ for any k_j. On the other hand, for any $k \in \mathbb{N}$, there exists $k_j \in \mathbb{N}$ such that $k_j \geq k$. Therefore, as the sequence $(\|f^k(x)\|)_{k\in\mathbb{N}}$ is decreasing (Lemma 2.2), we obtain that $\|f^k(x)\| \geq \|f^{k_j}(x)\| \geq \|y^0\|$.

We show now that $y^0 = 0$. Suppose the contrary, i.e. $y^0 \neq 0$.

Inequality (2.14) becomes

$$\|f^k(x)\| \geq \|y^0\| > 0 \qquad \text{for any } k \in \mathbb{N} \tag{2.15}$$

By means of Lemma 2.2, we have that $\|f(y^0)\| < \|y^0\|$.

Therefore, there exists a neighborhood $U_{f(y^0)} \subset B(R)$ of $f(y^0)$ such that for any $z \in U_{f(y^0)}$ we have $\|z\| < \|y^0\|$. On the other hand, for the neighborhood $U_{f(y^0)}$ there exists a neighborhood $U_{y^0} \subset B(R)$ of y^0 such that for any $y \in U_{y^0}$, we have $f(y) \in U_{f(y^0)}$. Therefore:

$$\|f(y)\| < \|y^0\| \qquad \text{for any } y \in U_{y^0} \tag{2.16}$$

As $f^{k_j}(x) \to y^0$, there exists $\bar{j}$ such that $f^{k_j}(x) \in U_{y^0}$, for any $j \geq \bar{j}$. Making $y = f^{k_j}(x)$ in (2.16), it results that

$$\|f^{k_j+1}(x)\| = \|f(f^{k_j}(x))\| < \|y^0\| \qquad \text{for } j \geq \bar{j} \tag{2.17}$$

which contradicts (2.15). This means that $y^0 = 0$, consequently, every convergent subsequence of $(f^k(x))_{k\in\mathbb{N}}$ converges to 0. This provides that the sequence $(f^k(x))_{k\in\mathbb{N}}$ is convergent to 0, and $x \in D_a(0)$.

Therefore, the ball $B(R)$ is contained in the domain of attraction of $D_a(0)$. □

For $p \geq 0$ and $c > 0$ let be N_p^c the set

$$N_p^c = \{x \in \Omega : V_p(x) < c\} \tag{2.18}$$

If $c = +\infty$, then $N_p^c = \Omega$.

Theorem 2.5. *Let be $p \geq 0$. For any $c \in (0, (p+1)R^2]$, the set N_p^c is included in the domain of attraction $D_a(0)$.*

Proof. Let be $c \in (0, (p+1)R^2]$ and $x \in N_p^c$. Then $V_p(x) = \sum_{k=0}^{p} \|f^k(x)\|^2 < c \leq (p+1)R^2$, therefore, there exists $k \in \{0, 1, .., p\}$ such that $\|f^k(x)\|^2 < R^2$. It results that $f^k(x) \in B(R) \subset D_a(0)$, therefore, $x \in D_a(0)$. □

Remark 2.6. *It is obvious that for $p \geq 0$ and $0 < c' < c''$ one has $N_p^{c'} \subset N_p^{c''}$. Therefore, for a given $p \geq 0$, the largest part of $D_a(0)$ which can be found by this method is $N_p^{c_p}$, where $c_p = (p+1)R^2$. In the followings, we will use the notation N_p instead of $N_p^{c_p}$. Shortly, $N_p = \{x \in \Omega : V_p(x) < (p+1)R^2\}$ is a part of $D_a(0)$. Let's note that $N_0 = B(R)$.*

Remark 2.7. *If $R = +\infty$ (i.e. $\Omega = \mathbb{R}^n$ and $\|f(x)\| < \|x\|$, for any $x \in \mathbb{R} \setminus \{0\}$), then $N_p = \mathbb{R}^n$ for any $p \geq 0$ and $D_a(0) = \mathbb{R}^n$.*

Theorem 2.8. *For the sets $(N_p)_{p\in\mathbb{N}}$, the following properties hold:*

a. *For any $p \geq 0$, one has $N_p \subset N_{p+1}$;*
b. *For any $p \geq 0$ the set N_p is invariant to f;*
c. *For any $x \in D_a(0)$ there exists $p^x \geq 0$ such that $x \in N_{p^x}$.*

Proof. a. Let be $p \geq 0$ and $x \in N_p$. Then $V_p(x) = \sum\limits_{k=0}^{p} \|f^k(x)\|^2 < (p+1)R^2$, therefore, there exists $k \in \{0, 1, .., p\}$ such that $\|f^k(x)\|^2 < R^2$. It results that $f^k(x) \in B(R)$ and therefore $f^m(x) \in B(R)$, for any $m \geq k$. For $m = p+1$ we obtain $\|f^{p+1}(x)\| < R$, hence $V_{p+1}(x) = V_p(x) + \|f^{p+1}(x)\|^2 < (p+1)R^2 + R^2 = (p+2)R^2$. Therefore, $x \in N_{p+1}$.

b. Let be $x \in N_p$. If $\|x\| < R$ then $\|f^m(x)\| < R$ for any $m \geq 0$ (by means of Lemma 2.2). This implies that $V_p(f(x)) = \sum\limits_{k=0}^{p} \|f^k(f(x))\|^2 = \sum\limits_{k=1}^{p+1} \|f^k(x)\|^2 < (p+1)R^2$, meaning that $f(x) \in N_p$.

Let's suppose that $\|x\| \geq R$. As $x \in N_p$, we have that $V_p(x) = \sum\limits_{k=0}^{p} \|f^k(x)\|^2 < (p+1)R^2$, therefore, there exists $k \in \{0, 1, .., p\}$ such that $\|f^k(x)\| < R$. It results that $f^k(x) \in B(R)$ and therefore $f^m(x) \in B(R)$, for any $m \geq k$. For $m = p+1$ we obtain $\|f^{p+1}(x)\| < R$. This implies that

$$V_p(f(x)) = V_p(x) + \|f^{p+1}(x)\|^2 - \|x\|^2 < (p+1)R^2 + R^2 - R^2 = (p+1)R^2 \quad (2.19)$$

therefore $f(x) \in N_p$.

c. Suppose the contrary, i.e. there exist $x \in D_a(0)$ such that for any $p \geq 0$, $x \notin N_p$. Therefore, $V_p(x) \geq (p+1)R^2$ for any $p \geq 0$. Passing to the limit for $p \to \infty$ in this inequality, provides that $V(x) = \infty$. This means $x \in \partial D_a(0)$ which contradicts the fact that x belongs to the open set $D_a(0)$. In conclusion, there exists $p^x \geq 0$ such that $x \in N_{p^x}$. □

For $p \geq 0$ let be $M_p = f^{-p}(B(R)) = \{x \in \Omega : f^p(x) \in B(R)\}$, obtained by the trajectory reversing method.

Theorem 2.9. *The following properties hold:*

a. *$M_p \subset D_a(0)$ for any $p \geq 0$;*
b. *For any $p \geq 0$, M_p is invariant to f;*
c. *$M_p \subset M_{p+1}$ for any $p \geq 0$;*
d. *For any $x \in D_a(0)$ there exists $p^x \geq 0$ such that $x \in M_{p^x}$.*

Proof. a. As $M_p = f^{-p}(B(R))$ and $B(R) \subset D_a(0)$ (see Theorem 2.4) it is clear that $M_p \subset D_a(0)$.

b. and *c.* follow easily by induction, using Lemma 2.2.

d. $x \in D_a(0)$ provides that $f^p(x) \to 0$ as $p \to \infty$. Therefore, there exists $p^x \in \mathbb{N}$ such that $f^p(x) \in B(R)$, for any $p \geq p^x$. This provides that $x \in M_p$ for any $p \geq p^x$. □

Both sequences of sets $(M_p)_{p\in\mathbb{N}}$ and $(N_p)_{p\in\mathbb{N}}$ are increasing, and are made up of estimates of $D_a(0)$. From the practical point of view, it is important to know which sequence converges more quickly. The next theorem provides that the sequence $(M_p)_{p\in\mathbb{N}}$ converges more quickly than $(N_p)_{p\in\mathbb{N}}$, meaning that for $p \geq 0$, the set M_p is a larger estimate of $D_a(0)$ then N_p.

Theorem 2.10. *For any $p \geq 0$ one has $N_p \subset M_p$.*

Proof. Let be $p \geq 0$ and $x \in N_p$. We have that $V_p(x) = \sum\limits_{k=0}^{p} \|f^k(x)\|^2 < (p+1)R^2$, therefore, there exists $k \in \{0, 1, .., p\}$ such that $\|f^k(x)\| < R$. This implies that $f^m(x) \in B(R)$, for any $m \geq k$. For $m = p$ we obtain $f^p(x) \in B(R)$, meaning that $x \in M_p$. $\square$

3. Theoretical results when $A = \partial_0 f$ is a convergent matrix, i.e. $\rho(A) < 1$.

As $\rho(A) \leq \|A\|$, the case when A is a convergent matrix [4] is more general then the case when A is a contraction treated in the previous section.

Proposition 3.1. *If $\rho(A) < 1$, then there exists $\tilde{p} \in \mathbb{N}^\star$ and $r_{\tilde{p}} > 0$ such that $B(r_{\tilde{p}}) \subset \Omega$ and $\|f^p(x)\| < \|x\|$ for any $p \in \{\tilde{p}, \tilde{p}+1, .., 2\tilde{p}-1\}$ and $x \in B(r_{\tilde{p}}) \setminus \{0\}$.*

Proof. We have that $\rho(A) < 1$ if and only if $\lim\limits_{p\to\infty} A^p = 0$ (see [4]), which provides that there exists $\tilde{p} \in \mathbb{N}^\star$ such that $\|A^p\| < 1$ for any $p \geq \tilde{p}$. Let be $\tilde{p} \in \mathbb{N}^\star$ fixed with this property.

The formula of variation of constants for any p gives:

$$f^p(x) = A^p x + \sum_{k=0}^{p-1} A^{p-k-1} g(f^k(x)) \qquad \text{for all } x \in \Omega \text{ and } p \in \mathbb{N}^\star \tag{3.20}$$

Due to the fact that for any $k \in \mathbb{N}$ we have $\lim\limits_{x\to 0} \frac{\|g(f^k(x))\|}{\|x\|} = 0$, there exists $r_{\tilde{p}} > 0$ such that for any $p \in \{\tilde{p}, \tilde{p}+1, .., 2\tilde{p}-1\}$ the following inequality holds:

$$\sum_{k=0}^{p-1} \|A^{p-k-1}\| \|g(f^k(x))\| < (1 - \|A^p\|)\|x\| \qquad \text{for } x \in B(r_{\tilde{p}}) \setminus \{0\} \tag{3.21}$$

Let be $x \in B(r_{\tilde{p}}) \setminus \{0\}$ and $p \in \{\tilde{p}, \tilde{p}+1, .., 2\tilde{p}-1\}$. Using (3.20) and (3.21) we have

$$\begin{aligned} \|f^p(x)\| &= \|A^p x + \sum_{k=0}^{p-1} A^{p-k-1} g(f^k(x))\| \leq \\ &\leq \|A^p\|\|x\| + \sum_{k=0}^{p-1} \|A^{p-k-1}\|\|g(f^k(x))\| < \\ &< (\|A^p\| + 1 - \|A^p\|)\|x\| = \|x\| \end{aligned} \tag{3.22}$$

Therefore, $\|f^p(x)\| < \|x\|$ for $p \in \{\tilde{p}, \tilde{p}+1, .., 2\tilde{p}-1\}$ and $x \in B(r_{\tilde{p}}) \setminus \{0\}$. $\square$

Definition 3.1. *Let be $\tilde{p} \in \mathbb{N}^\star$ the smallest number such that $\|A^p\| < 1$ for any $p \geq \tilde{p}$ (see the proof of Proposition 3.1). Let be $\tilde{R} > 0$ the largest number such that $B(\tilde{R}) \subset \Omega$ and $\|f^p(x)\| < \|x\|$ for $p \in \{\tilde{p}, \tilde{p}+1, .., 2\tilde{p}-1\}$ and $x \in B(\tilde{R}) \setminus \{0\}$.*

If for any $r > 0$ we have that $B(r) \subset \Omega$ and $\|f^p(x)\| < \|x\|$ for any $p \in \{\tilde{p}, \tilde{p}+1, .., 2\tilde{p}-1\}$ and $x \in B(r) \setminus \{0\}$, then $\tilde{R} = +\infty$ and $B(\tilde{R}) = \Omega = \mathbb{R}^n$.

Lemma 3.2.
a. *For any $x \in B(\tilde{R})$ and $p \in \{\tilde{p}, \tilde{p}+1, .., 2\tilde{p}-1\}$ the sequence $(\|f^{kp}(x)\|)_{k\in\mathbb{N}}$ is decreasing.*
b. *For any $p \geq \tilde{p}$ and $x \in B(\tilde{R}) \setminus \{0\}$, $\|f^p(x)\| < \|x\|$.*
c. *For any $p \geq \tilde{p}$ and $x \in B(\tilde{R}) \setminus \{0\}$, $\Delta V_p(x) = V_p(f(x)) - V_p(x) < 0$, where V_p is defined by (2.13).*

Proof. a. If $x = 0$, then $f^p(0) = 0$, for any $p \geq 0$.

Let be $x \in B(\tilde{R}) \setminus \{0\}$. We know that $\|f^p(x)\| < \|x\|$ for any $p \in \{\tilde{p}, \tilde{p}+1, .., 2\tilde{p}-1\}$. It results that $f^p(x) \in B(\tilde{R})$ for any $p \in \{\tilde{p}, \tilde{p}+1, .., 2\tilde{p}-1\}$. This implies that for any $k \in \mathbb{N}^\star$ we have $\|f^{kp}(x)\| < \|x\|$ and $\|f^{(k+1)p}(x)\| \leq \|f^{kp}(x)\|$, meaning that the sequence $(\|f^{kp}(x)\|)_{k\in\mathbb{N}}$ is decreasing.

b. Let be $x \in B(\tilde{R}) \setminus \{0\}$. Inequality $\|f^p(x)\| < \|x\|$ is true for any $p \in \{\tilde{p}, \tilde{p}+1, .., 2\tilde{p}-1\}$.

Let be $p \geq 2\tilde{p}$. There exists $q \in \mathbb{N}^\star$ and $p' \in \{\tilde{p}, \tilde{p}+1, .., 2\tilde{p}-1\}$ such that $p = q\tilde{p} + p'$. Using *a.*, we have that $f^{p'}(x) \in B(\tilde{R})$ and $f^{q\tilde{p}}(y) \leq \|y\|$, for any $y \in B(\tilde{R})$, therefore

$$\|f^p(x)\| = \|f^{q\tilde{p}}(f^{p'}(x))\| \leq \|f^{p'}(x)\| < \|x\| \tag{3.23}$$

c. results directly from *b.*

□

Corollary 3.3. *For any $p \geq \tilde{p}$, there exists a maximal domain $G_p \subset \Omega$ such that $0 \in G_p$ and for any $x \in G_p \setminus \{0\}$, the (positive definite) function V_p verifies $\Delta V_p(x) < 0$. In other words, for any $p \geq \tilde{p}$ the function V_p is a Lyapunov function for (1.1) on G_p. More, $B(\tilde{R}) \subset G_p$ for any $p \geq \tilde{p}$.*

Lemma 3.4. *For any $k \geq \tilde{p}$ there exists $q_k \in \mathbb{N}$ such that*

$$\|f^{(q_k+3)\tilde{p}}(x)\| \leq \|f^k(x)\| \leq \|f^{q_k\tilde{p}}(x)\| \qquad \textit{for any } x \in B(\tilde{R}) \tag{3.24}$$

Proof. Let be $k \geq \tilde{p}$. There exists a unique $q_k \in \mathbb{N}$ and a unique $p_k \in \{\tilde{p}, \tilde{p}+1, .., 2\tilde{p}-1\}$ such that $k = q_k\tilde{p} + p_k$. Lemma 3.2 provides that for any $x \in B(\tilde{R})$ we have that $f^{q_k\tilde{p}}(x) \in B(\tilde{R})$ and $\|f^{p_k}(x)\| \leq \|x\|$. It results that

$$\|f^k(x)\| = \|f^{p_k}(f^{q_k\tilde{p}}(x))\| \leq \|f^{q_k\tilde{p}}(x)\| \qquad \text{for any } x \in B(\bar{R}) \tag{3.25}$$

On the other hand, we have $(q_k+3)\tilde{p} = k+(3\tilde{p}-p_k)$. As $(3\tilde{p}-p_k) \in \{\tilde{p}+1, \tilde{p}+2, .., 2\tilde{p}\}$ and $k \geq \tilde{p}$, Lemma 3.2 provides that for any $x \in B(\tilde{R})$ we have that $f^k(x) \in B(\tilde{R})$ and $\|f^{3\tilde{p}-p_k}(x)\| \leq \|x\|$. Therefore

$$\|f^{(q_k+3)\tilde{p}}(x)\| = \|f^{3\tilde{p}-p_k}(f^k(x))\| \leq \|f^k(x)\| \qquad \text{for any } x \in B(\tilde{R}) \tag{3.26}$$

Combining the two inequalities, we get that

$$\|f^{(q_k+3)\tilde{p}}(x)\| \leq \|f^k(x)\| \leq \|f^{q_k\tilde{p}}(x)\| \qquad \text{for any } x \in B(\tilde{R}) \tag{3.27}$$

which concludes the proof. □

Theorem 3.5. *$B(\tilde{R})$ is included in the domain of attraction $D_a(0)$.*

Proof. Let be $x \in B(\tilde{R}) \setminus \{0\}$. We have to prove that $\lim_{k\to\infty} f^k(x) = 0$.

The sequence $(f^k(x))_{k\in\mathbb{N}}$ is bounded (see Lemma 3.2). Let be $(f^{k_j}(x))_{j\in\mathbb{N}}$ a convergent subsequence and let be $\lim_{j\to\infty} f^{k_j}(x) = y^0$.

We suppose, without loss of generality, that $k_j \geq \tilde{p}$ for any $j \in \mathbb{N}$. Lemma 3.4 provides that for any $j \in \mathbb{N}$ there exists $q_j \in \mathbb{N}$ such that

$$\|f^{(q_j+3)\tilde{p}}(x)\| \leq \|f^{k_j}(x)\| \leq \|f^{q_j\tilde{p}}(x)\| \tag{3.28}$$

As $(\|f^{q_j\tilde{p}}(x)\|)_{j\in\mathbb{N}}$ and $(\|f^{(q_j+3)\tilde{p}}(x)\|)_{j\in\mathbb{N}}$ are subsequences of the convergent sequence $(\|f^{q\tilde{p}}(x)\|)_{q\in\mathbb{N}}$ (decreasing, according to Lemma 3.2), it results that they

are convergent. The double inequality (3.28) provides that $\lim_{j\to\infty} \|f^{q_j\tilde{p}}(x)\| = \|y^0\|$. Therefore, $\lim_{q\to\infty} \|f^{q\tilde{p}}(x)\| = \|y^0\|$.

It can be shown that

$$\|f^k(x)\| \geq \|y^0\| \qquad \text{for any } k \geq \tilde{p} \tag{3.29}$$

For this, remark that $\lim_{q\to\infty} \|f^{q\tilde{p}}(x)\| = \|y^0\|$ and $(\|f^{q\tilde{p}}(x)\|)_{q\in\mathbb{N}}$ is decreasing (Lemma 3.2), which implies that $\|f^{q\tilde{p}}(x)\| \geq \|y^0\|$ for any $q \in \mathbb{N}$. On the other hand, Lemma 3.4 provides that for any $k \geq \tilde{p}$ there exists q_k such that $\|f^{(q_k+3)\tilde{p}}(x)\| \leq \|f^k(x)\|$. Therefore, $\|f^k(x)\| \geq \|f^{(q_k+3)\tilde{p}}(x)\| \geq \|y^0\|$, for any $k \geq \tilde{p}$.

We show now that $y^0 = 0$. Suppose the contrary, i.e. $y^0 \neq 0$.

Inequality (3.29) becomes

$$\|f^k(x)\| \geq \|y^0\| > 0 \qquad \text{for any } k \geq \tilde{p} \tag{3.30}$$

By means of Lemma 3.2, we have that $\|f^{\tilde{p}}(y^0)\| < \|y^0\|$.

There exists a neighborhood $U_{f^{\tilde{p}}(y^0)} \subset B(\tilde{R})$ of $f^{\tilde{p}}(y^0)$ such that for any $z \in U_{f^{\tilde{p}}(y^0)}$ we have $\|z\| < \|y^0\|$. On the other hand, for the neighborhood $U_{f^{\tilde{p}}(y^0)}$ there exists a neighborhood $U_{y^0} \subset B(\tilde{R})$ of y^0 such that for any $y \in U_{y^0}$, we have $f^{\tilde{p}}(y) \in U_{f^{\tilde{p}}(y^0)}$. Therefore:

$$\|f^{\tilde{p}}(y)\| < \|y^0\| \qquad \text{for any } y \in U_{y^0} \tag{3.31}$$

As $f^{k_j}(x) \to y^0$, there exists $\bar{j}$ such that $f^{k_j}(x) \in U_{y^0}$, for any $j \geq \bar{j}$. Making $y = f^{k_j}(x)$ in (3.31), it results that

$$\|f^{k_j+\tilde{p}}(x)\| = \|f^{\tilde{p}}(f^{k_j}(x))\| < \|y^0\| \qquad \text{for } j \geq \bar{j} \tag{3.32}$$

which contradicts (3.30). This means that $y^0 = 0$, consequently, every convergent subsequence of $(f^k(x))_{k\in\mathbb{N}}$ converges to 0. This provides that the sequence $(f^k(x))_{k\in\mathbb{N}}$ is convergent to 0, and $x \in D_a(0)$.

Therefore, the ball $B(\tilde{R})$ is contained in the domain of attraction of $D_a(0)$. □

Theorem 3.6. *Let be $p \geq \tilde{p}$. For any $c \in (0, (p+1)\tilde{R}^2]$, the set N_p^c is included in the domain of attraction $D_a(0)$.*

Proof. Let be $c \in (0, (p+1)\tilde{R}^2]$ and $x \in N_p^c$. Then $V_p(x) = \sum_{k=0}^{p} \|f^k(x)\|^2 < c \leq (p+1)\tilde{R}^2$, therefore, there exists $k \in \{0, 1, .., p\}$ such that $\|f^k(x)\|^2 < \tilde{R}^2$. It results that $f^k(x) \in B(\tilde{R}) \subset D_a(0)$, therefore, $x \in D_a(0)$. □

Remark 3.7. *It is obvious that for $p \geq \tilde{p}$ and $0 < c' < c''$ one has $N_p^{c'} \subset N_p^{c''}$. Therefore, for a given $p \geq \tilde{p}$, the largest part of $D_a(0)$ which can be found by this method is $N_p^{\tilde{c}_p}$, where $\tilde{c}_p = (p+1)\tilde{R}^2$. In the followings, we will use the notation $\tilde{N}_p$ instead of $N_p^{\tilde{c}_p}$. Shortly, $\tilde{N}_p = \{x \in \Omega : V_p(x) < (p+1)\tilde{R}^2\}$ is a part of $D_a(0)$.*

Remark 3.8. *If $\tilde{R} = +\infty$ (i.e. $\Omega = \mathbb{R}^n$ and $\|f^p(x)\| < \|x\|$, for any $p \in \{\tilde{p}, \tilde{p}+1, .., 2\tilde{p}-1\}$ and $x \in \mathbb{R} \setminus \{0\}$), then $\tilde{N}_p = \mathbb{R}^n$ for any $p \geq \tilde{p}$ and $D_a(0) = \mathbb{R}^n$.*

Theorem 3.9. *For any $x \in D_a(0)$ there exists $p^x \geq \tilde{p}$ such that $x \in \tilde{N}_{p^x}$.*

Proof. Let be $x \in D_a(0)$. Suppose the contrary, i.e. $x \notin \tilde{N}_p$ for any $p \geq \tilde{p}$. Therefore, $V_p(x) \geq (p+1)\tilde{R}^2$ for any $p \geq \tilde{p}$. Passing to the limit when $p \to \infty$ in this inequality provides that $V(x) = \infty$. This means $x \in \partial D_a(0)$ which contradicts

the fact that x belongs to the open set $D_a(0)$. In conclusion, there exists $p^x \geq 0$ such that $x \in \tilde{N}_{p^x}$. □

Remark 3.10. *In the case when $\|\partial_0 f\| < 1$ we have shown that the sequence of sets $(N_p)_{p\in\mathbb{N}}$ is increasing (see Theorem 2.8).*

Open Question: *If $\|\partial_0 f\| \geq 1$ and $\rho(\partial_0 f) < 1$, is the sequence of sets $(\tilde{N}_p)_{p\in\mathbb{N}}$ increasing?*

For $p \geq 0$ let be $\tilde{M}_p = f^{-p}(B(\tilde{R})) = \{x \in \Omega : f^p(x) \in B(\tilde{R})\}$, obtained by the trajectory reversing method.

Theorem 3.11. *For the sets $(\tilde{M}_p)_{p\in\mathbb{N}}$ the following properties hold:*

a. $\tilde{M}_p \subset D_a(0)$, *for any* $p \geq \tilde{p}$;
b. $\tilde{M}_{kp} \subset \tilde{M}_{(k+1)p}$ *for any* $k \in \mathbb{N}$ *and* $p \in \{\tilde{p}, \tilde{p}+1, .., 2\tilde{p}-1\}$;
c. *For any* $x \in D_a(0)$ *there exists* $p^x \geq \tilde{p}$ *such that* $x \in \tilde{M}_{p^x}$.

Proof. a. As $\tilde{M}_p = f^{-p}(B(\tilde{R}))$ and $B(\tilde{R}) \subset D_a(0)$ (see Theorem 3.5) it is clear that $\tilde{M}_p \subset D_a(0)$.

b. follows easily by induction, using Lemma 3.2.

c. $x \in D_a(0)$ provides that $f^p(x) \to 0$ as $p \to \infty$. Therefore, there exists $p^x \geq \tilde{p}$ such that $f^p(x) \in B(\tilde{R})$, for any $p \geq p^x$. This provides that $x \in \tilde{M}_p$ for any $p \geq p^x$. □

Remark 3.12. *The sequence of sets $(\tilde{M}_p)_{p\in\mathbb{N}}$ is generally not increasing (see Section 4: Numerical examples, the Van der Pol equation).*

Both sequences of sets $(\tilde{M}_p)_{p\in\mathbb{N}}$ and $(\tilde{N}_p)_{p\in\mathbb{N}}$ are made up of estimates of $D_a(0)$. From the practical point of view, it would be important to know which one of the sets $\tilde{M}_p$ or $\tilde{N}_p$ is a larger estimate of $D_a(0)$ for a fixed $p \geq \tilde{p}$. Such result could not be established, but the following theorem holds:

Theorem 3.13. *For any $p \geq 0$ one has $\tilde{N}_p \subset \tilde{M}_{p+\tilde{p}}$.*

Proof. Let be $p \geq 0$ and $x \in \tilde{N}_p$. We have that $V_p(x) = \sum\limits_{k=0}^{p} \|f^k(x)\|^2 < (p+1)\tilde{R}^2$, therefore, there exists $k \in \{0, 1, .., p\}$ such that $\|f^k(x)\| < \tilde{R}$. This implies that $f^{k+m}(x) \in B(\tilde{R})$, for any $m \geq \tilde{p}$. For $m = p - k + \tilde{p}$ we obtain $f^{p+\tilde{p}}(x) \in B(\tilde{R})$, meaning that $x \in \tilde{M}_{p+\tilde{p}}$. □

4. **Numerical examples.**

4.1. **Example with known domain of attraction.**

Let be the following discrete dynamical system

$$\begin{cases} x_{k+1} = \frac{1}{2}x_k(1 + x_k^2 + 2y_k^2) \\ y_{k+1} = \frac{1}{2}y_k(1 + x_k^2 + 2y_k^2) \end{cases} \qquad k \in \mathbb{N} \tag{4.33}$$

There exists an infinity of steady states for this system: $(0,0)$ (asymptotically stable) and all the points (x,y) belonging to the ellipsis $x^2 + 2y^2 = 1$ (all unstable). The domain of attraction of $(0,0)$ is $D_a(0,0) = \{(x,y) \in \mathbb{R}^2 : x^2 + 2y^2 < 1\}$.

As $\|\partial_{(0,0)} f\| = \frac{1}{2}$, we compute the largest number $R > 0$ such that $\|f(x)\| < \|x\|$ for any $x \in B(R) \setminus \{0\}$, and we find $R = 0.7071$.

For $p = 0, 1, 2, 3, 4$ we find the N_p sets shown in Figure 1, parts of $D_a(0, 0)$ ($N_p \subset N_{p+1}$, for $p \geq 0$). In Figure 1, the thick-contoured ellipsis represents the boundary of $D_a(0, 0)$.

In Figure 2, the sets M_p are represented, for $p = \overline{0, 6}$ ($M_p \subset M_{p+1}$, for $p \geq 0$). Note that M_6 approximates with a good accuracy the domain of attraction.

4.2. **Discrete predator-prey system.** We consider the discrete predator-prey system:

$$\begin{cases} x_{k+1} = ax_k(1 - x_k) - x_k y_k \\ y_{k+1} = \frac{1}{b} x_k y_k \end{cases} \quad \text{with } a = \frac{1}{2}, b = 1, k \in \mathbb{N} \tag{4.34}$$

The steady states of this system are: $(0, 0)$ (asymptotically stable), $(-1, 0)$ and $(1, -1)$ (both unstable).

We have that $\|\partial_{(0,0)} f\| = \frac{1}{2}$, and the largest number $R > 0$ such that $\|f(x)\| < \|x\|$ for any $x \in B(R) \setminus \{0\}$ is $R = 0.65$.

Figure 3 presents the N_p sets for $p = 0, 1, 2, 3, 4, 5$, parts of $D_a(0, 0)$ ($N_p \subset N_{p+1}$, for $p \geq 0$). The black points in Figure 3 represent the steady states of the system.

In Figure 4, the sets M_p are represented, for $p = \overline{0, 6}$ ($M_p \subset M_{p+1}$, for $p \geq 0$). Note that the boundary of M_6 approaches very much the fixed points $(-1, 0)$ and $(1, -1)$, which suggests that M_6 is a good approximation of $D_a(0)$.

4.3. **Discrete Van der Pol system.** Let be the following discrete dynamical system, obtained from the continuous Van der Pol system:

$$\begin{cases} x_{k+1} = x_k - y_k \\ y_{k+1} = x_k + (1 - a)y_k + a x_k^2 y_k \end{cases} \quad \text{with } a = 2, k \in \mathbb{N} \tag{4.35}$$

The only steady state of this system is $(0, 0)$ which is asymptotically stable. There are many periodic points for this system, the periodic points of order $\overline{2, 5}$ being represented in Figure 5 by the black points.

We have that $\|\partial_{(0,0)} f\| = 2$ but $\rho(\partial_{(0,0)} f) = 0$. First, we observe that for $\tilde{p} = 2$ we have that $(\partial_{(0,0)} f)^{\tilde{p}} = O_2$, therefore, $\|(\partial_{(0,0)} f)^p\| = 0$ for any $p \geq \tilde{p}$.

The largest number $\tilde{R} > 0$ such that $\|f^p(x)\| < \|x\|$ for $p \in \{\tilde{p}, \tilde{p}+1, .., 2\tilde{p}-1\} = \{2, 3\}$ and $x \in B(\tilde{R}) \setminus \{0\}$ is $\tilde{R} = 0.365$.

For $p = 2, 3, 4, 5$, the connected components which contain $(0, 0)$ of the $\tilde{N}_p$ sets are shown in Figure 5. We have that $\tilde{N}_2 \subset \tilde{N}_3 \subset \tilde{N}_4 \subset \tilde{N}_5$.

In Figure 6, the sets $\tilde{M}_p$ are represented, for $p = \overline{0, 6}$. Note that the inclusion $\tilde{M}_p \subset \tilde{M}_{p+1}$ does not hold.

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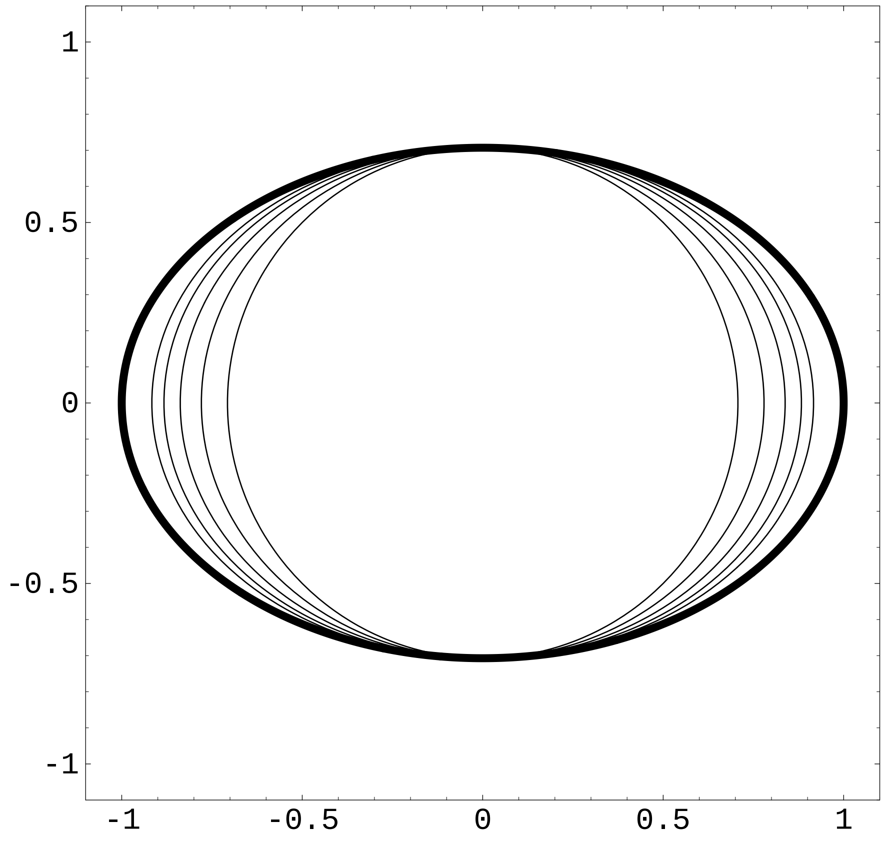

FIGURE 1. The sets N_p, $p = \overline{0,4}$ and $\partial D_a(0,0)$ for (4.33)

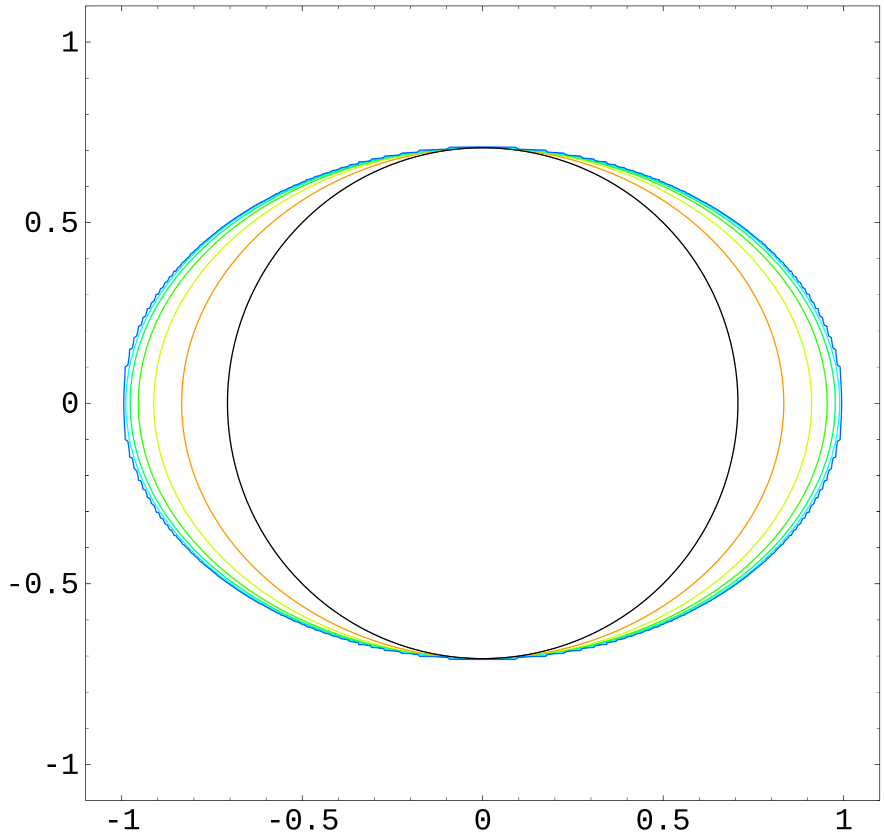

FIGURE 2. The sets M_p, $p = \overline{0,6}$ for (4.33)

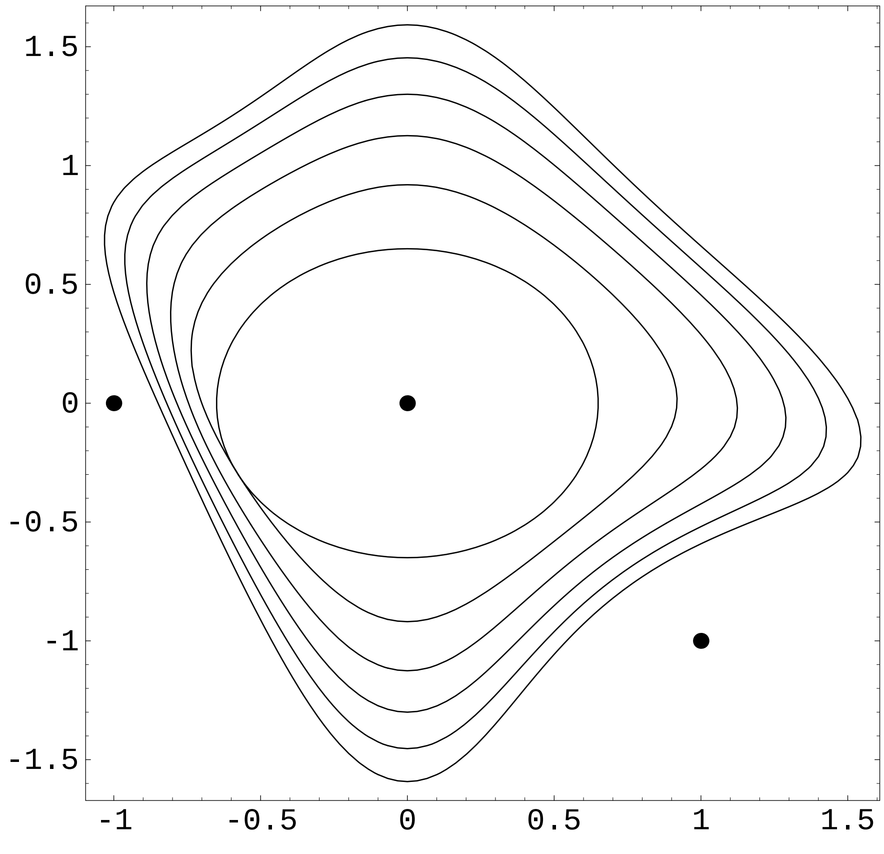

FIGURE 3. The sets N_p, $p = \overline{0,5}$ for (4.34)

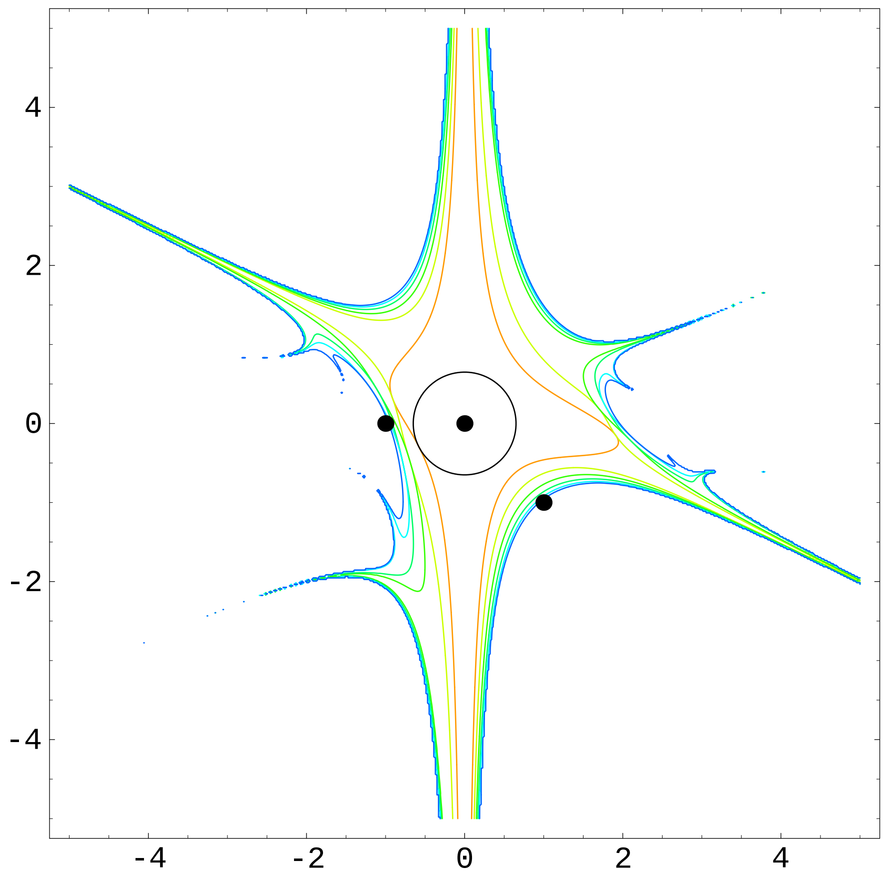

FIGURE 4. The sets M_p, $p = \overline{0, 6}$ for (4.34)

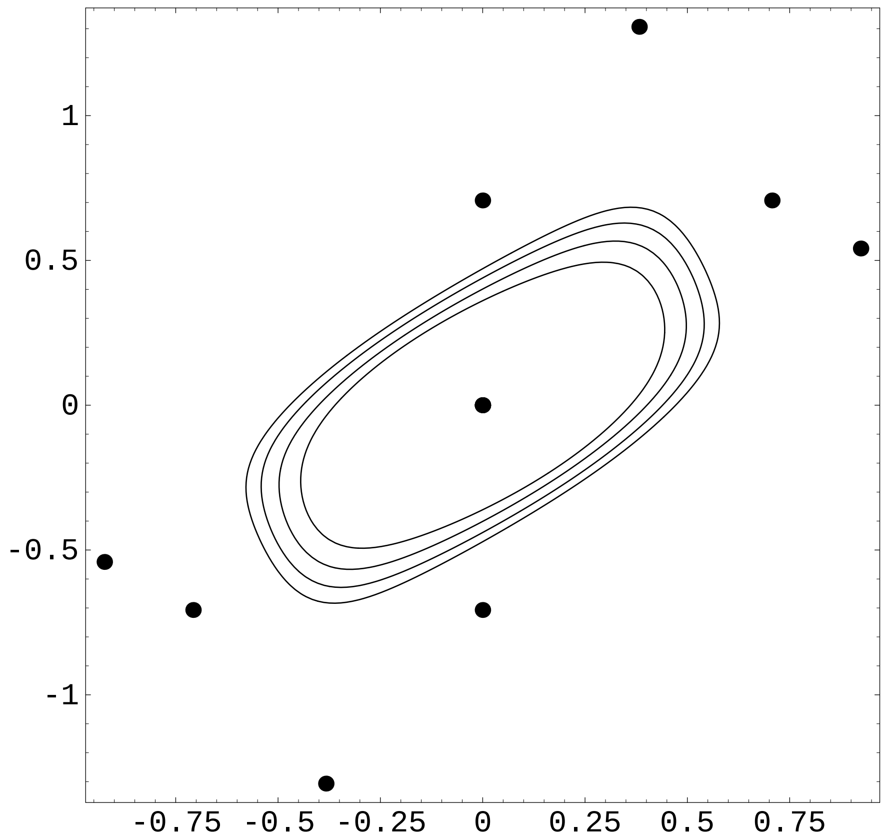

FIGURE 5. The sets $\tilde{N}_p$, $p = \overline{2,5}$ for (4.35)

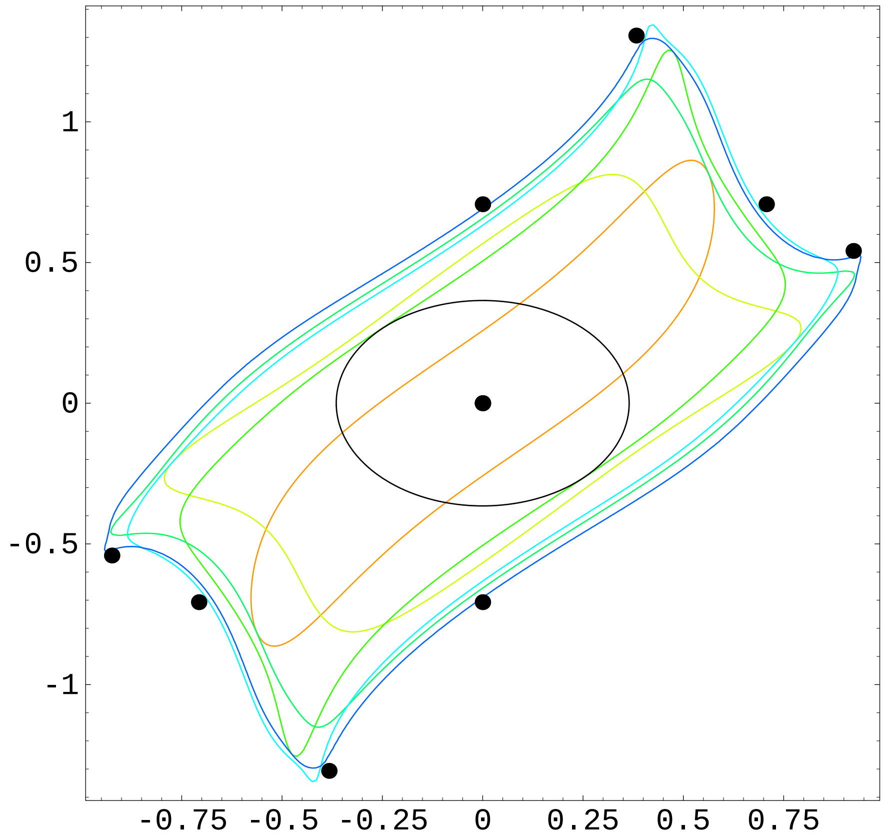

FIGURE 6. The sets $\tilde{M}_p$, $p = \overline{0,6}$ for (4.35)

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